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REAL EIGENVALUES OF VISCOELASTIC OSCILLATORS INVOLVING SEVERAL DAMPING MODELS

ALEKSANDRA KOSTIĆ AND HEINRICH VOSS

ABSTRACT. This paper presents the dynamic analysis of a system involving various damping models. The free motion equation can be transformed to a system containing only exponentially decaying damping. Its real eigenvalues can be characterized as maxmin values of a Rayleigh functional and can be determined by the quadratically convergent saveguarded iteration. A numerical example demonstrates the efficiency of the approach.

1. INTRODUCTION

The nature of energy dissipation mechanisms in a vibrating structure have always been very difficult to explain: damping models have been developed by trying to fit experimental and mathematical results. The viscous approach proposed by Lord Rayleigh assuming that dissipative forces are proportional to the velocity of the systems degrees of freedom is the damping model used for the great majority of structural solid materials (metal, concrete, wood, glass, etc.)

The weakness of the pure viscous model becomes evident when applying it to the so called viscoelastic materials such as polymer derivatives and rubbers or rubber like materials which are characterized by a time-dependent constitutive model and by frequency dependent Young's and shear moduli. The viscoelastic damping is introduced into the system assuming that the dissipative forces are proportional to the history of the velocity via kernel hereditary functions.

For small displacements the most general form of a viscoelastic damped oscillator becomes

$$\mathbf{M}\ddot{\mathbf{u}}(t)+\int_0^t \mathcal{G}(t-\tau)\dot{\mathbf{u}}(\tau)d\tau+\mathbf{K}\mathbf{u}(t)=\mathbf{f}(t), \tag{1.1}$$

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together with initial conditions $\mathbf{u}(0) = \mathbf{u_0}$, $\dot{\mathbf{u}}(0) = \dot{\mathbf{u}}_0$, where $\mathbf{u} \in \mathbb{R}^N$ is the displacement vector, $\mathbf{f} \in \mathbb{R}^N$ is the forcing vector, $\mathbf{M} \in \mathbb{R}^{N\times N}$ and $\mathbf{K} \in \mathbb{R}^{N\times N}$ are the mass and stiffness matrix, respectively, and $\mathcal{G} \in \mathbb{R}^{N\times N}$ is the kernel function of damping.

The modes of the system can be determined as non-trivial solutions of the free-motion problem. With functions of the form $\mathbf{u}(t) = \mathbf{u}e^{st}$ we get

$$\mathbf{T}(s) := \left(s^2\mathbf{M} + s\left(\mathbf{C}_0 + \sum_{k=1}^{n} G_k(s)\mathbf{C}_k\right) + \mathbf{K}\right)\mathbf{u} = 0, \tag{1.2}$$

where $\mathbf{C}_0 \in \mathbb{R}^{N\times N}$ is the frequency–independent viscous damping matrix, $\mathbf{C}_k \in \mathbb{R}^{N\times N}$ are the coefficient matrices of frequency dependent damping, and $G_k(s)$ are the frequency dependent non–viscous damping functions.

In practice, mechanical engineering systems with two or more parts with significant different levels of energy dissipation are encountered frequently in dynamical design, so these damping systems often involve multiple damping models.

Many different damping models have been proposed to describe the dissipative behavior of viscoelastic materials. We mention some available damping models from the literature: The Biot model [3]

$$sG_k(s) = a_k s/(s+b_k),$$

the anelastic displacement field model [7]

$$sG_k(s) = \Delta_k s/(s+\Omega_k),$$

and the generalized Maxwell model [13]

$$sG_k(s) = \frac{K_k\gamma_k s}{\gamma_k s + K_k}.$$

For all of them the term $sG_k(s)C_k$ appearing in (1.2) can be transformed equivalently to $s\tilde{G}_k(s)\tilde{C}_k$ where

$$\tilde{G}_k(s) := \frac{\mu_k}{s+\mu_k} \tag{1.3}$$

is the exponential damping model [1].

A further rational damping model $sG_k(s) = \frac{\alpha_k s^2 + \gamma_k s}{s^2 + \beta_k s + \delta_k}$ introduced by Golla and Hughes [5] can be transformed with $\alpha_k = a_1 + a_2$, $\gamma_k = a_1 b_2 + a_2 b_1$, $\beta_k = b_1 + b_2$, $\delta_k = b_1 b_2$ to the sum of two Biot models, and then further to two exponential models.

Li and Hu [8] aggregated the rational terms of different damping models in one fraction formula of a rational polynomial

$$G(s) = \sum_{j=0}^{p_k} c_j s^j \Bigg/ \sum_{j=0}^{q_k} d_j s^j$$

and studied the damping model (1.2) taking advantage of the theory of minimal realization [2]. In this paper we rewrite every damping model as an exponential damping model and study the linear viscoelastic system in the following form

$$\mathbf{T}(s)\mathbf{u} := (s^2\mathbf{M} + s\mathbf{C_0} + s\mathbf{G}(s) + \mathbf{K})\mathbf{u} = 0, \tag{1.4}$$

where all $G_k s$ are rewritten to damping models of exponential type (1.3).

$$\mathbf{G}(s) = \sum_{j=1}^{n} \frac{\mu_j}{s+\mu_j}\mathbf{C}_j. \tag{1.5}$$

Since its non–real eigenvalues can be determined from its corresponding undamped system by Neuman expansion we consider only the real eigenvalues. In a similar way as in [9, 10] we provide a variational characterization, which allows for an efficient numerical computation by safeguarded iteration. A numerical example demonstrates the effectiveness of this approach.

2. Variational characterization for nonlinear eigenvalue problems

Our main tool for computing the real eigenvalues of problem (1.4), (1.5) is the variational characterization of eigenvalues of nonlinear eigenvalue problems generalizing the well known maxmin characterization of Poincaré (1890) for linear eigenvalue problems.

We consider the nonlinear eigenvalue problem

$$\mathbf{T}(s)\mathbf{x} = 0, \tag{2.1}$$

where $\mathbf{T}(s) \in \mathbb{C}^{n\times n}$, $s \in J$, is a family of Hermitian matrices depending continuously on the parameter $s \in J$, and J is a real open interval which may be unbounded.

To generalize the variational characterization of eigenvalues we need a generalization of the Rayleigh quotient. To this end we assume that

(A_1) for every fixed $\mathbf{x} \in \mathbb{C}^n$, $\mathbf{x} \neq 0$ the scalar real equation

$$f(s;x) := \mathbf{x}^H\mathbf{T}(s)\mathbf{x} = 0 \tag{2.2}$$

has at most one solution $p(\mathbf{x}) \in J$.

Then $f(s;\mathbf{x}) = 0$ implicitly defines a functional p on some subset $\mathcal{D} \subset \mathbb{C}^n$ which is called the Rayleigh functional of problem (2.1), and which is exactly the Rayleigh quotient in case of a monic linear matrix function $\mathbf{T}(s) := s\mathbf{I} - \mathbf{A}$.

Generalizing the definiteness requirement for linear pencils $\mathbf{T}(s) = s\mathbf{B} - \mathbf{A}$ we further assume that

(A_2) for every $\mathbf{x} \in \mathcal{D}$ and every $s \in J$ with $s \neq p(\mathbf{x})$ it holds that

$$(s - p(\mathbf{x}))\, f(s;\mathbf{x}) > 0. \tag{2.3}$$

If p is defined on $\mathcal{D} = \mathbb{C}^n \setminus \{0\}$ then the problem $\mathbf{T}(s)\mathbf{x} = 0$ is called overdamped. Generalizations of the minmax and maxmin characterizations of eigenvalues were proved by Duffin (1955) for the quadratic case and by Rogers (1964) for general overdamped problems.

For nonoverdamped eigenproblems the natural ordering to call the largest eigenvalue the first one, the second largest the second one, etc., is not appropriate. This is obvious if we make a linear eigenvalue problem $\mathbf{T}(s)\mathbf{x} := (s\mathbf{I} - \mathbf{A})\mathbf{x} = 0$ nonlinear by restricting it to an interval J which does not contain the largest eigenvalue of $\mathbf{A}$. Then the conditions (A_1) and (A_2) are satisfied, p is the restriction of the Rayleigh quotient R_A to $\mathcal{D} := \{\mathbf{x} \neq 0 : R_A(\mathbf{x}) \in J\}$, and $\sup_{\mathbf{x}\in\mathcal{D}} p(\mathbf{x})$ will in general not be an eigenvalue.

We introduce the following numeration of the eigenvalues: If $s \in J$ is an eigenvalue of $\mathbf{T}(\cdot)$ then $\mu = 0$ is an eigenvalue of the linear problem $\mathbf{T}(s)\mathbf{y} = \mu\mathbf{y}$, and therefore there exists $\ell \in \mathbb{N}$ such that

$$0 = \min_{V\in H_\ell} \max_{\mathbf{v}\in V\setminus\{0\}} \frac{\mathbf{v}^H\mathbf{T}(s)\mathbf{v}}{\|\mathbf{v}\|^2}$$

where H_ℓ denotes the set of all ℓ–dimensional subspaces of $\mathbb{C}^n$. In this case s is called an ℓth eigenvalue of $T(\cdot)$.

With this numeration the following maxmin characterization holds which was proved in Voss and Werner [16] and in the more general form given here in [15]. These two papers are generalizations of the maxmin characterization of Poincraré [12] for linear eigenvalue problems and by Duffin [4] for overdamped quadratic problems, and by Rogers [14] for the general overdamped case.

Theorem 2.1. *Let J be an open interval in $\mathbb{R}$, and let $\mathbf{T}(s) \in \mathbb{C}^{n\times n}$, $s \in J$, be a family of Hermitian matrices depending continuously on the parameter $s \in J$ such that the conditions (A_1) and (A_2) are satisfied. Then the following statements hold.*

(i) *For every $\ell \in \mathbb{N}$ there is at most one ℓth eigenvalue of $\mathbf{T}(\cdot)$ in J which can be characterized by*

$$s_\ell = \max_{V\in H_\ell,\ V\cap\mathcal{D}\neq\emptyset} \inf_{\mathbf{v}\in V\cap\mathcal{D}} p(\mathbf{v}). \tag{2.4}$$

(ii) *If*

$$s_\ell := \sup_{V\in H_\ell,\ V\cap\mathcal{D}\neq\emptyset} \inf_{\mathbf{v}\in V\cap\mathcal{D}} p(\mathbf{v}) \in J$$

for some $\ell \in \mathbb{N}$ then s_ℓ is the ℓth eigenvalue of $T(\cdot)$ in J, and (2.4) *holds.*

(iii) *If there exist the kth and the ℓth eigenvalue s_k and s_ℓ in J ($k > \ell$), then J contains the jth eigenvalue s_j ($k \geq j \geq \ell$) as well, and $s_k \leq s_j \leq s_\ell$.*

(iv) *Let $s_1 = \sup_{x\in\mathcal{D}} p(\mathbf{x}) \in J$ and $s_\ell \in J$. If the maximum in* (2.4) *is attained for an ℓ dimensional subspace V, then $V \subset \mathcal{D}\cup\{0\}$, and* (2.4) *can be replaced with*

$$s_\ell = \max_{V\in H_\ell,\ V\subset\mathcal{D}\cup\{0\}} \min_{\mathbf{v}\in V,\ \mathbf{v}\neq 0} p(\mathbf{v}). \tag{2.5}$$

The number of eigenvalues in an interval can be determined from the following generalization of Sylvester's law of inertia for nonlinear eigenvalue problems which was proved in [6].

Theorem 2.2. *Let* $\mathbf{T} : J \to \mathbb{C}^{n\times n}$ *satisfy the conditions of the maxmin characterization and let* (n_+, n_-, n_0) *be the inertia of* $\mathbf{T}(\sigma)$ *for some* $\sigma \in J$.

(i) *If* $\sup_{\mathbf{x}\in\mathcal{D}} p(\mathbf{x}) \in J$ *then the nonlinear eigenvalue problem* $\mathbf{T}(\lambda)\mathbf{x} = 0$ *has exactly* n_- *eigenvalues* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n_-}$ *in* J *not exceeding* σ.

(ii) *Let* $\sigma,\tau \in J$, $\sigma < \tau$, *and let* $(n_+(\sigma)n_-(\sigma), n_0(\sigma))$ *and* $(n_+(\tau), n_-(\tau), n_0(\tau))$ *be the inertia of* $\mathbf{T}(\sigma)$ *and* $\mathbf{T}(\tau)$, *respectively. Then the inequality* $n_-(\sigma) \geq n_-(\tau)$ *holds , and the nonlinear eigenvalue problem* $\mathbf{T}(\lambda)\mathbf{x} = 0$ *has exactly* $n_-(\sigma) - n_-(\tau)$ *eigenvalues* $\lambda_{n_-(\sigma)+1} \geq \cdots \geq \lambda_{n_-(\tau)}$ *in the interval* (σ,τ).

3. Real eigenvalues of viscoelastic systems

In the following we consider the viscoelastic linear system (1.4), (1.5) We assume that the mass matrix $\mathbf{M}$ and the stiffness matrix $\mathbf{K}$ are positive definite, that the damping coefficient matrices $\mathbf{C}_j \neq 0$, $j = 0,\dots,n$ are positive semi–definite, and that the relaxation parameters are ordered by magnitude $0 < \mu_1 < \cdots < \mu_n$.

By r_j we denote the rank of the matrix $\mathbf{C}_j$ and the jth interval by $I_j := (-\mu_j, -\mu_{j-1})$, $j = 1,\dots,n$ where $\mu_0 := 0$.

For every $\mathbf{u} \in \mathbb{R}^N \setminus \{0\}$ we consider the projection of problem (1.4) to span$\{\mathbf{u}\}$

$$\begin{aligned} f(s;\mathbf{u}) &:= s^2\mathbf{u}^T\mathbf{M}\mathbf{u} + s\mathbf{u}^T\mathbf{C}_0\mathbf{u} + s\sum_{j=1}^{n} \frac{\mu_j}{s+\mu_j}\mathbf{u}^T\mathbf{C}_j\mathbf{u} + \mathbf{u}^T\mathbf{K}\mathbf{u} \\ &=: ms^2 + sc_0 + s\sum_{j=1}^{n}\frac{\mu_j}{s+\mu_j}c_j + k = 0, \end{aligned} \tag{3.1}$$

which is the characteristic equation of a single–degree–of–freedom system.

If each of these systems is underdamped, i.e. two of its eigenvalues are in a complex conjugate pair, then the eigenvalue problem (1.4), (1.5) has $m = 2N + \sum_{j=1}^n r_j$ eigenvalues, $r := \sum_{j=1}^n r_j$ of which are real due to the underdamping assumption [17].

The elastic modes corresponding to non–real eigenvalues can be obtained easily from the eigenset $(\mathbf{x}_j, \omega_j^2)$, $j = 1,\dots,N$ of the undamped system $\mathbf{K}\mathbf{x} = \omega^2\mathbf{M}\mathbf{x}$ by the Neumann expansion method to any desired order of accuracy (cf. Adhikari [1], p. 193). In the following we therefore consider only the characterization of the real eigenvalues and its numerical computation.

We first show that all eigenvalues in I_j, $j \in \{1,\dots,N\}$, are maxmin values of the Rayleigh functional p_j corresponding to $\mathbf{T}(\cdot)$ on the interval I_j. For fixed $\mathbf{u}$ assume that $c_j := \mathbf{u}^T\mathbf{C}_j\mathbf{u} \neq 0$ for $j = j_1 < j_2 < \cdots < j_\ell$, $\ell \leq n$ and consider the one–dimensional system

$$f(s;\mathbf{u}) = s^2m + sc_0 + s\sum_{j=1}^{n}\frac{\mu_j}{s+\mu_j}c_j + k = s^2m + sc_0 + s\sum_{i=1}^{\ell}\frac{\mu_{j_i}}{s+\mu_{j_i}}c_{j_i} + k.$$

Then $f(\cdot;\mathbf{u})$ has ℓ poles $\mu_{j_1},\dots,\mu_{j_\ell}$ each of order 1, and since $f(s;\mathbf{u}) = 0$ has $\ell+2$ roots, two of which are non–real, it has exactly one root in each of the intervals $(-\mu_{j_i}, -\mu_{j_{i-1}})$ for $i = 1,\dots,\ell$, where $j_0 = 0$. Hence, $f(\cdot;\mathbf{u})$ has at most one root in I_j for every $j \in \{1,\dots,n\}$, and the Rayleigh functional p_j corresponding to I_j is defined. Moreover,

$$\lim_{s\to-\mu_{j_i}+0} s\frac{\mu_{j_i}}{s+\mu_{j_i}}\mathbf{u}^T\mathbf{C}_{j_i}\mathbf{u} = -\infty, \quad \text{hence} \lim_{s\to-\mu_{j_i}+0} f(s;\mathbf{u}) = -\infty,$$

and

$$\lim_{s\to-\mu_{j_{i-1}}-0} s\frac{\mu_{j_{i-1}}}{s+\mu_{j_{i-1}}}\mathbf{u}^T\mathbf{C}_{j_{i-1}}\mathbf{u} = \infty, \quad \text{hence} \lim_{s\to-\mu_{j_{i-1}-0}} f(s;\mathbf{u}) = \infty,$$

and therefore condition (A_2) is also satisfied.

The number of eigenvalues in I_j is obtained from the generalization of Sylvester's law Theorem 2.2:

Theorem 3.1. *Assume that the general conditions on* $\mathbf{M}$, $\mathbf{K}$, *and* $\mu_j,\ j = 1,..,n$ *are satisfied, and assume that all matrices* $\mathbf{C}_j$ *are positive semi–definite.*

Let $(n_+(s), n_-(s), n_0(s))$ *be the inertia of* $\mathbf{T}(s)$. *Then the following characterizations of eigenvalues in* I_j *of* $\mathbf{T}(\cdot)$ *holds.*

Let $-\mu_j < \sigma < \tau < -\mu_{j-1}$. *Then* $\mathbf{T}(\cdot)$ *has* $n_-(\sigma) - n_-(\tau)$ *eigenvalues* $s^{(j)}_{n_-\sigma+1} \geq \cdots \geq s^{(j)}_{n_-\tau}$ *in the interval* $(\sigma,\tau]$ *which can be characterized as*

$$s^{(j)}_\ell = \max_{V\cap\mathcal{D}_j\neq\emptyset, \dim V=\ell} \ \inf_{\mathbf{u}\in V\cap\mathcal{D}_j} p_j(\mathbf{u}),\ \ell = n-\sigma+1,\dots,n-\tau.$$

For the first the interval $I_1 = (-\mu_1, 0)$ we even get

Theorem 3.2. *Under the general conditions of Theorem 3.1 the interval* $I_1 := (-\mu_1, 0)$ *contains (at least)* $n_-(-\mu_1+\varepsilon) \geq r_1$ *eigenvalues, and all eigenvalues* $s^{(1)}_1 \geq s^{(1)}_2 \geq \cdots$ *can be characterized as*

$$s^{(1)}_\ell = \max_{V\subset\mathcal{D}_1, \dim V=\ell} \ \min_{\mathbf{u}\in V} p_1(\mathbf{u}).$$

Proof. We only have to show, that $\sup_{\mathbf{u}\in\mathcal{D}_1} p_1(\mathbf{u}) < 0$, which follows from

$$\frac{\partial}{\partial s} f(s;\mathbf{u})\Big|_{s=0} = \mathbf{u}^T\mathbf{C}_0\mathbf{u} + \sum_{j=1}^{n}\mathbf{u}^T\mathbf{C}_j\mathbf{u} > 0,$$

and that $\mathcal{D}_1$ contains an r_1 dimensional subset, which follows from

$$\lim_{s\to-\mu_1+0} s\frac{\mu_1}{s+\mu_1}\mathbf{u}^T\mathbf{C}_1\mathbf{u} = -\infty, \quad \text{i.e.} \lim_{s\to-\mu_1+0} f(s;\mathbf{u}) = -\infty,$$

and $\mathbf{u}^T\mathbf{T}(0)\mathbf{u} = \mathbf{u}^T\mathbf{K}\mathbf{u} > 0$ for every $\mathbf{u}$ such that $\mathbf{u}^T\mathbf{C}_1\mathbf{u} > 0$. □

The proof of the maxmin characterization Theorem 2.1 reveals that the subspace for which the maximum in (2.4) is attained is the invariant subspace of $\mathbf{T}(s_j)$ which is spanned by the eigenvectors of the matrix $\mathbf{T}(s_j)$ corresponding to its jth smallest eigenvalue, and that the minimum is attained for every eigenvector of $\mathbf{T}(s_j)\mathbf{u} = \kappa\mathbf{u}$ corresponding to its eigenvalue $\kappa = 0$. This suggests the following method called safeguarded iteration for computing the jth eigenvalue of $\mathbf{T}(\cdot)$.

Safeguarded iteration was shown to be globally convergent for a first eigenvalue and quadratically convergent for simple eigenvalues [11]. It is quite inexpensive, because in each step one has to determine a particular eigenvector of the linear symmetric eigenvalue problem $\mathbf{T}(\sigma_{k-1})\mathbf{v} = \kappa\mathbf{v}$ and to solve the real equation $\mathbf{u}_k^H\mathbf{T}(\sigma_k)\mathbf{v}_k = 0$ for σ_k.

Algorithm 1 Safeguarded iteration

Require: initial vector $\mathbf{v}_0 \in \mathcal{D}$
1: compute $\sigma_0 = p(\mathbf{v}_0)$
2: **for** $k = 1, 2, \dots$ until convergence **do**
3: determine an eigenvector $\mathbf{v}_k$ corresponding to the j smallest eigenvalue of $\mathbf{T}(\sigma_{k-1})$
4: determine Rayleigh functional $\sigma_k := p(\mathbf{v}_k)$, i.e. solve $\mathbf{v}_k^H\mathbf{T}(\sigma_k)\mathbf{v}_k = 0$ for σ_k
5: **end for**

4. Numerical example

Consider the visoelastic eigenvalue problem (1.2) with $N = 3$, $n = 4$ and

$$\mathbf{M} = 3\mathbf{I},\ \mathbf{K} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix},\ \mathbf{C}_0 = \mathbf{O}$$

$$\mathbf{C}_1 = 0.4\begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \mathbf{C}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \mathbf{C}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$G_1(s) = \frac{2}{2+s},\ G_2(s) = \frac{3}{3+s},\ G_3(s) = \frac{5s^2+8s}{s^2+5s+4}.$$

The function $G_3(s)$ can be rewritten as

$$G_3(s) = \frac{5s^2+8s}{s^2+5s+4} = \frac{s}{1+s} + \frac{4s}{4+s}$$

which implies that the problem is equivalent to

$$T(s)\mathbf{u} = \left(s^2\mathbf{M} + \frac{s}{1+s}\mathbf{C}_3 + \frac{2s}{s+2}\mathbf{C}_1 + \frac{3s}{3+s}\mathbf{C}_2 + \frac{4s}{s+4}\mathbf{C}_3 + \mathbf{K}\right)\mathbf{u} = 0,$$

which has five real eigenvalues, at least one in $(-1,0)$, three in $(-2,0)$, four in $(-3,0)$, and five in $(-4,0)$.

Actually, the problem has two eigenvalues in $(-1,0)$, two in $(-2,-1)$, no eigenvalue in $(-3,-2)$, and one eigenvalue in $(-4,-3)$. Safeguarded iteration yields for the largest eigenvalue

j	λ_j	rel. error
0	-0.5	1.30 E-1
1	-0.449	1.51 E-2
2	-0.442653	1.85 E-4
3	-0.44257201	2.70 E-8
4	-0.4425719998646988	

which demonstrates the quadratic convergence.

The further real eigenvalues are

j	λ_j
2	-0.70677554203146
3	-1.16855466952597
4	-1.84800062899244
5	-3.44165006484647

which are also obtained by safeguarded iteration with quadratic convergence.

5. Conclusions

Mechanical engineering systems frequently contain different parts with significantly different levels of energy dissipation which may involve different damping models. This paper demonstrates that the corresponding free motion equation often can be rewritten to a form that exhibits only exponentially decaying damping. Its real eigenvalues can be characterized as maxmin values of a Rayleigh functional and can be determined efficiently by the quadratically convergent safeguarded iteration. A numerical example demonstrates the behavior of the method.

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Aleksandra Kostić
Mašinski fakultet Sarajevo
Univerzitet Sarajevo
Bosnia and Herzegovina
kostic@mef.unsa.ba

Heinrich Voss
Institute of Mathematics
Hamburg University of Technology
D-21073 Hamburg
Germany
voss@tuhh.de